

REMOVABILITY:
FIRST ORDER SOBOLEV SPACE,
PDE SOLUTIONS, HOLOMORPHIC FUNCTIONS.
– SOBOLEV SPACE CONSTRUCTED IN A NEW WAY.

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ABSTRACT. Conditions for removability for sets with respect to first order Sobolev space functions, holomorphic functions and other solutions to first order PDE:s – also nonlinear – are proved. A new kind of representation formula for first order Sobolev space is proved. This result can also be used as a new definition of these spaces and is useful for the discussion of removability of sets in \mathbf{R}^N with respect to this space.

The problem of removability for first order Sobolev space gives answers to a question on the classification of Julia sets, see [DOA-HUB].

This treatment of Sobolev space removability is a follow up of a paper by Peter Jones, see [JON].

1. INTRODUCTION AND DISCUSSION

First it should be said that this paper is a follow up of a paper by Peter Jones, [JON], and that the main contribution here is Theorem 2.16, which gives a new way to define first order Sobolev space, more about this later in this section.

The main part of the present paper have been given as work document, [WAN], and this have been sent to Peter Jones some years ago. This have had effect on his later work on the problem. Peter Jones and Stanislav Smirnov have recently produced a paper, [JON-SMI].

After this has been said we begin a longer introduction/discussion to give background and motivation.

In a general way of speaking a removability problem can be formulated as follows. Let \mathcal{F} and \mathcal{G} be classes of functions from say \mathbf{R}^N to say \mathbf{R} or \mathbf{C} . Suppose that this class \mathcal{G} is a nice class e.g. the holomorphic functions on the Riemann sphere \mathbf{C} or the real-valued Sobolev functions in $W^{m,p}(\mathbf{R}^N)$.

Suppose that \mathcal{F} has the same kind of description as that of \mathcal{G} but with a deficit in domain so to speak, e.g. are holomorphic on the complement of a set S or are Sobolev functions on the complement of S . Then suppose there are extra conditions as well – this is often the case.

The question of removability in these settings is to find conditions that implies that the set S can be removed, i.e. all functions in \mathcal{F} are restrictions of functions in \mathcal{G} . Negative answers are of course interesting as well.

Next we go through some removability problems. For the moment let's assume Sobolev space to be known with some definition or the other. Proper definitions will be given later.

Notation 1.1. Let Ω be an open subset of \mathbf{R}^N and let $W^{1,p}(\Omega)$ denote Sobolev space of order one with exponent p and with domain Ω .

In order to get nicer formulas we also introduce an adjusted notation.

Notation 1.2. Let Ω be open in \mathbf{R}^N . Denote with $W^{1,p}(\Omega, \mathbf{R}^N)$ the set of functions from \mathbf{R}^N to \mathbf{R} , which have their restriction to Ω in $W^{1,p}(\Omega)$.

Now we choose to discuss removability in terms of compacts rather than closed sets in order to conform with the formulations in [JON].

Let $C(\mathbf{R}^N)$ denote the space of real continuous functions on \mathbf{R}^N .

Definition 1.3. Let $p \geq 1$, let K be compact in \mathbf{R}^N . Then put $K = \Omega^c$. That K has the J_p -property if the following two subspaces are equal

$$(1.1) \quad C(\mathbf{R}^N) \cap W^{1,p}(\Omega, \mathbf{R}^N) = C(\mathbf{R}^N) \cap W^{1,p}(\mathbf{R}^N).$$

Note that the inclusion RHS \subset LHS in (1.1) is obvious and (1.1) can be reduced to LHS \subset RHS.

Hence that a compact has the J_p -property has the meaning that the compact is removable in $W^{1,p}$ with continuity as extra property.

Next we introduce some removability problems involving holomorphic functions.

For some of these there is a background involving capacity concepts. There are two standard capacities for analytic functions, analytic capacity $\gamma(K)$ and continuous analytic capacity $\alpha(K)$. For definitions and results concerning these see the book by Garnett, [GAR].

Theorem. If ω is an open subset of \mathbf{C} and compact K subset of ω then the property that all u continuous in the ω and holomorphic in $\omega \setminus K$ also are holomorphic in the whole ω holds if and only if $\alpha(K) = 0$.

We can give some of the contents of this theorem without giving the definition of $\alpha(K)$.

Theorem. Besicovitch, see [BES]. If K is compact in \mathbf{C} and if K is a countable union of sets, each with finite one-dimensional Hausdorff measure, then $\alpha(K) = 0$.

Hence $\alpha(K) = 0$ implies that K is removable in the sense given.

Next we give another removability problem for holomorphic functions.

Definition 1.4. Let K be a compact subset of the Riemann sphere \mathbf{C} . The set K is said to have the *HR*-property (Holomorphically Removable) if that F is a homeomorphism from \mathbf{C} to \mathbf{C} with F holomorphic off K implies that F is a Möbius transformation.

That this is a removability property follows from that if F is a Möbius transformation then K was removable and on the other hand if K is removable then F is defined everywhere as an analytic function on the Riemann sphere and then that F also is a homeomorphism implies that F is a Möbius transformation.

Hence if K is removable in the sense related to continuous analytic capacity then K has the *HR*-property. Hence also $\alpha(K) = 0$ implies that K has the *HR*-property.

One background for the interest in the *HR*-property is found in the classical work by Doady and Hubbard, [DOA-HUB], on iteration of one complex variable quadratic polynomials.

Doady and Hubbard proved among many things that the corresponding Julia sets can be divided into disjoint classes in a natural way and that such a class of Julia sets in this classification has exactly one member if and only if any Julia set in the class has the *HR*-property.

We follow Jones and identify \mathbf{C} and \mathbf{R}^2 , see [JON]. This way the compacts can be seen as subset of both and the *HR*-property and the J_p -property can be compared.

The results by Peter Jones in [JON] can be summed up as follows. Here (i) is an observation and the main part is (ii).

- (i) The J_2 -property implies the *HR*-property.
- (ii) The boundary of a John domain has the J_2 -property.

The definition of John domains is well-known. We refer to [JON].

This result may seem strange at first, since it is obvious that very simple domains, e.g. a domain that form a simple cusp, have the J_2 -property and they are clearly not John domains.

The explanation is that Peter Jones was posed exactly the question to prove the *HR*-property for the boundary of a general John domain in order to give an answer to a question concerning a special situation in Dynamical System theory.

However from the function space point of view the correct setting of this removability problem is with the weaker property quasicontinuity instead of continuity. The quasicontinuity property will be defined in the next section.

We define other kinds of removability akin to the previous ones.

It is natural to think about these different definitions of removability and what implications there are between them.

- (i) The continuous J_p -property. This is written $J_p(\text{cont})$.
- (ii) The quasicontinuous J_p -property. Written $J_p(\text{quasicont})$ or just simply J_p .
- (iii) The J_p and $J_p(\text{cont})$ properties locally in the sense of L^p -norms is written $J_{p,\text{loc}}$ and $J_{p,\text{loc}}(\text{cont})$.
- (iv) It follows that J_p implies $J_p(\text{cont})$.
- (v) It follows that $J_{p,\text{loc}} = J_p$ and $J_{p,\text{loc}}(\text{cont}) = J_p(\text{cont})$ by the use of partition of unity.
- (vi) HL_p , $HL_p(\text{cont})$, $HL_{p,\text{loc}}$, etc. defined as removability of holomorphic functions with respect to the L^p -norm of the gradient and otherwise in accordance with the definition of J_p etc.

Some open questions.

- I:** Is there a compact K such that J_p does not hold for K but $J_p(\text{cont})$ holds.
- II:** Is there a compact K such that J_p , ($J_p(\text{cont})$), does not hold for K , but holds for J_q , ($J_q(\text{cont})$), $p \neq q$.
- III:** Jones: Is there a compact K such that HR holds for K but not for $J_2(\text{cont})$.

A discussion on results in general.

A problem is not solved just because there exists a proved necessary and sufficient condition. The real issue at stake is how many of the interesting cases are covered and can be tested. There is here also a question about what type of descriptions, geometrical, topological etc. that can be covered.

- These or similar kinds of questions are also essential for other fields of problems. It would be profitable if such questions could be more (publicly) discussed.

In our case there are plenty of results on removability though it is for the moment not possible to reach as far into the tough cases as wanted, compare here [JON] and [JON-SMI].

Since $J_p(\text{cont})$ is simpler to study the main effort is spent on the more difficult J_p case.

– It is interesting that by posing a question in this function space-wise correct way we are lead to make a new, alternative and interesting definition of first order Sobolev space.

Next we formulate a well-known fact, which will be important. Since the proof is short it is included.

Theorem 1.5. Let u be a function on \mathbf{R}^N , let u be absolutely continuous on almost all lines in the x_1 -direction. Let $u, D_1 u \in L^1_{\text{loc}}$, then the weak derivative $D_1 u$ exists and is given as the pointwise derivative $D_1 u$.

Proof. On each of these special lines the derivative $D_1 u$ exists a.e. by the absolute continuity, but this is a.a. lines, hence the pointwise derivative $D_1 u$ is defined a.e. in the domain.

However the pointwise derivative can be identified with the weak derivative a distribution.

Simply take a test function φ and evaluate the action of $D_1 u$ on φ as usual.

$$(1.2) \quad - \int u D_1 \varphi dx = - \int \int_L u D_1 \varphi dt dx' = \int \int_L D_1 u \varphi dt dx',$$

by partial integration, which works since u is absolute continuous on allmost all lines L .

End of Proof.

Definition 1.6. Let $V^{1,p}(\Omega)$ be what could be called the rough Sobolev space on the domain Ω in \mathbf{R}^N . This definition is that both the function and gradient is pointwise defined a.e. Hence the corresponding Sobolev space norm makes sense. It should be finite as well.

– In the same way as is done in Notation 1.2 we assume that the functions in $V^{1,p}(\mathbf{R}^N)$ are defined in all \mathbf{R}^N .

The main result, Theorem 2.16.

There are two equivalent formulations. Here we discuss (2.11).

In words (2.11) states that the rough Sobolev space $V^{1,p}(\Omega)$ when intersected with the the set of quasicontinuous functions gives a direct sum in the vector space sense.

The first summand is in fact the ordinary Sobolev space but with functions defined in a precise sense quasi everywhere. The second summand contains the the singular part of this subspace of the rough Sobolev space also in a precise sense.

How does this apply to the J_p -property?

Let the open set Ω in the formula be K^c . Let $u \in W^{1,p}(K^c)$. Now it is not difficult to see that a necessary condition for J_p to hold is that the Lebesgue measure of K is zero. – Hence assume this. It follows then that $u \in V^{1,p}(\mathbf{R}^N)$.

Then study the J_p -property or the $J_p(\text{cont})$ -property.

– In either case the property is true if and only if the second direct summand is the empty set.

Corollary 1.7. The J_p -property, ($J_p(\text{cont})$ -property), for compact K in \mathbf{R}^N holds if and only if there does not exist a function $f \in Q \cap V^{1,p}(\mathbf{R}^N)$, ($f \in C \cap V^{1,p}(\mathbf{R}^N)$), such that $f \in W^{1,p}(K^c)$ and f is singular on a set of parallel lines of nonzero $(N-1)$ -dim Lebesgue measure.

Corollary 1.8. It is necessary for the J_p property to be untrue that there is a set of parallel lines of nonzero $(N - 1)$ -dim Lebesgue measure that intersect K uncountably many times, since a singular function on a line always has uncountable support.

Corollary 1.9. It is sufficient for the J_p property to hold that there are N non-degenerate directions such that the lines in these directions intersect K with a $(N - 1)$ -dim Lebesgue measure for these lines are equal to zero.

Removability with respect to PDO:s.

Given a function u which is the weak solution to a first order PDE in $\mathbf{R}^N \setminus S$. Where S is closed and of Lebesgue measure zero. Suppose that $D_1 u$ and u are in L^1 . Then by Theorem 1.5 if a.a. lines in this direction intersect S countably many times then $D_1 u$ is a global weak derivative. This can be done in any direction. Also say linear complex combinations can be proven to give say $\bar{\partial} u$ as global weak derivative.

Observe. The PDO does not have be linear above.

Remark. Hence $J_2(\text{cont})$ implies $HL_2(\text{cont})$, which implies HR . (Cf. area theorem.)

Though we knew this before from [JON].

2. MAIN SECTION

To begin with some notations and definitions.

Definition 2.1. Let Ω be an open set in \mathbf{R}^N and let $T(\Omega)$ be the set of measurable functions from Ω to \mathbf{R} , then the L^p -norm of a function in $u \in T(\Omega)$ is

$$(2.1) \quad \|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u|^p dx \right)^{1/p}.$$

The corresponding first order Sobolev space norm is defined as follows.

Definition 2.2.

$$(2.2) \quad \|u\|_{W^{1,p}(\Omega)} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{L^p(\Omega)},$$

where α is a multiindex. It has always to be clear what definition of derivative that is used, since this can change.

Comment 2.3. There are other definitions of first order Sobolev space norms. They are usually equivalent (in the usual analysis sense) to the one given above.

Our way to get at the promised new point on first order Sobolev space is by several definitions of similar spaces, different super- and subspaces. Hence many more definitions will be needed.

First we give the Sobolev definition of Sobolev space. It was however not the first one. It is an early example of the use of weak derivatives long before elaborate distribution theory.

Definition 2.4. Let Ω be open subset of \mathbf{R}^N , then

$$(2.3) \quad W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \|u\|_{W^{1,p}(\Omega)} < \infty\},$$

where derivatives are taken in the weak sense.

Definition 2.5. Let Ω be open subset of \mathbf{R}^N , then the rough Sobolev space is defined as

$$(2.4) \quad V^{1,p}(\Omega) = \{u \in L^p(\Omega) : \|u\|_{W^{1,p}(\Omega)} < \infty\}.$$

The derivatives are taken in the usual sense and exist a.e.

We assume the concept of absolute continuity to be familiar.

Definition 2.6. Denote by $[AC]_j$ the subset of $T(\Omega)$ of members that they are absolutely continuous with respect to almost all $(N-1)$ -dimensional Lebesgue measure) lines parallel with the x_j axis in any closed cube parallel to the axii and subset of Ω . Furthermore denote

$$(2.5) \quad [AC] = \bigcap_{j=1}^N [AC]_j$$

The following are important and in fact show how much Sobolev space is related to L^p -space and this not only formally.

Definition 2.7. Let u be a function from Ω open in \mathbf{R}^N to \mathbf{R} . Let \mathcal{F} be a family of subsets of Ω that contains the open subsets. Let Ψ a mapping from \mathcal{F} to the nonnegative real numbers. The family \mathcal{F} is now denoted \mathcal{F}_Ψ .

The function u is said to be quasi continuous on Ω with respect to Ψ , if to every $\epsilon > 0$ there exists an open ω subset of Ω such that $\Psi(\omega) < \epsilon$ and $u|_{\Omega \setminus \omega}$ is continuous.

It is well-known that in Sobolev space the place of Lebesgue measure is taken by a set function called capacity i.e. in this case a very special capacity. Often the definition of this capacity is taken from potential theory and more specifically non-linear potential theory, i.e. when $p \neq 2$.

We will use the intrinsic definition of capacity that comes from the definition of the Sobolev space. For $1 < p$ these definitions give equivalent values.

The capacity $C_{1,p}$, also written as $(1,p)$ -capacity, is a set function mapping a certain family of sets into the nonnegative reals.

C.f. Ψ above.

It is closely connected to the structure of the first order Sobolev space.

Definition 2.8. Let K be a compact set in \mathbf{R}^N . Then define

$$(2.6) \quad C_{1,p}(K) = \inf \left\{ \sum_{|\alpha| \leq 1} \|D^\alpha \varphi\|_{L^p(\mathbf{R}^N)} : \varphi \in C_0^\infty(\mathbf{R}^N), \varphi|_K \geq 1 \right\}.$$

The following steps serve the purpose to generalize the domain of definition for the capacity.

Definition 2.9. Let ω be an open set in \mathbf{R}^N . Then define

$$(2.7) \quad C_{1,p}(\omega) = \sup_K \{C_{1,p}(K) : K \text{ compact}, K \subset \omega\}$$

Now we do a presentation proper, though there is a possibility to make a short-cut in the presentation here.

Definition 2.10. A set $M \in \mathbf{R}^N$ is capacitable if

$$(2.8) \quad \inf\{C_{1,p}(\omega) : \omega \text{ open}, M \subset \omega\} = \sup\{C_{1,p}(K) : K \text{ compact}, K \subset M\}$$

The family of all capacitable subsets of \mathbf{R}^N is denoted \mathcal{F}_C .

Hence we let capacity be a mapping from \mathcal{F}_C to the nonnegative reals and with the above procedure as definition.

The capacity defined here has many properties. We will recall what is needed, when needed.

Yet we say the following. In the first order case it follows by so called truncation that it is enough to consider nonnegative φ :s in the first step in the definition of $(1,p)$ -capacity above. It is then obvious that this capacity is subadditive, cf. measures that are additive.

For higher order Sobolev space this question is dealt with non-linear potential theory.

Next we have some facts, well-known from non-linear potential theory. This theory assumes in our case that $p > 1$.

Fact 2.11. The space $W^{1,p}(\mathbf{R}^N)$ equals a disjunct union of sets of functions, which are equal almost everywhere within each such set. There is in each set at least one member, which is quasi continuous with respect to $C_{1,p}$.

Definition 2.12.

$$(2.9) \quad Q = (Q^{1,p}(\Omega)) = \{u \in T(\Omega) : u \text{ is quasi continuous w.r.t. } C_{1,p}\}$$

Furthermore:

Fact 2.13. Uniqueness: Let $u, u' \in W^{1,p}(\mathbf{R}^N)$ and say that $u = u'$ almost everywhere and say furthermore that u and u' are quasi continuous (with respect $(1,p)$ -capacity) then $u = u'$ quasi everywhere, i.e. they differ on a set of $(1,p)$ -capacity zero only.

Definition 2.14. Let S_j be the subset of elements $T(\Omega)$ with j th partial derivative zero, as defined a.e.

Definition 2.15. Let \check{X}_j be the subset $T(\Omega)$ such that the elements are independent of x_j on almost all lines parallel to the j th axis.

The new point of view of (first order) Sobolev space is expressed by the equalities (2.10) and (2.11) below. It serves also as an alternative definition of this space.

Theorem 2.16. Let $1 \leq p$. Let the functions below be considered equal if they are equal quasi everywhere, i.e. differ only on a set of $(1,p)$ -capacity zero. The quotients and direct sum are here taken in the vector space sense, i.e. no topology concerns. Then it holds that

$$(2.10) \quad Q \cap V^{1,p}(\Omega) / \sum_{j=1}^N [S_j \cap Q \cap V^{1,p}(\Omega) / \check{X}_j \cap Q \cap V^{1,p}(\Omega)] = [AC] \cap Q \cap V^{1,p}(\Omega)$$

or equivalently

$$(2.11) \quad Q \cap V^{1,p}(\Omega) = [AC] \cap Q \cap V^{1,p}(\Omega) \oplus \sum_{j=1}^N [S_j \cap Q \cap V^{1,p}(\Omega) / \check{X}_j \cap Q \cap V^{1,p}(\Omega)].$$

Proof. It is given as a sequence of lemmas etc.

Lemma 2.17. Observe that from the vector space structure we have by Zorn's lemma a vector space basis B of $Q \cap V^{1,p}(\Omega)$. Since

$$(2.12) \quad \check{X}_j \cap Q \cap V^{1,p}(\Omega) \subset S_j \cap Q \cap V^{1,p}(\Omega) \subset Q \cap V^{1,p}(\Omega)$$

as subspaces, there furthermore exists such a basis B that there are $B_j \subset B$, with the following properties.

(i) Every B_j is a basis of

$$(2.13) \quad S_j \cap Q \cap V^{1,p}(\Omega) / \check{X}_j \cap Q \cap V^{1,p}(\Omega).$$

(ii) Let J be a subset of the whole numbers $1, 2, \dots, N$, then $\bigcup_{j \in J} B_j$ is a basis of

$$(2.14) \quad \sum_{j \in J} [S_j \cap Q \cap V^{1,p}(\Omega) / \check{X}_j \cap Q \cap V^{1,p}(\Omega)].$$

(iii) With J as above, $\bigcap_{j \in J} B_j$ is a basis of

$$(2.15) \quad \bigcap_{j \in J} S_j \cap Q \cap V^{1,p}(\Omega) / \check{X}_j \cap Q \cap V^{1,p}(\Omega).$$

The proof the statements (i)–(iii) follows simply by the following steps. First a partition of the original vector space in a Venn diagram way and an ordering of the corresponding “atomic” subspaces. Next by finitely many successive extensions of basis we get the properties needed.

It now follows from Lemma 2.17 that if such a basis is chosen then (2.10) and (2.11) are equivalent just by set operations on the elements in the basis.

Next we give some facts that will be needed in order to proceed.

Fact 2.18. The capacity $C_{1,p}(M) \geq m(M)$, where m is Lebesgue measure.

Proof: This follows from the definitions. Lebesgue measure m can be defined as the capacity $C_{0,p}$.

The possibility of the above definition of Lebesgue measure in fact is responsible for the aforementioned similarities between L^p and Sobolev space.

Fact 2.19. Let $C_{1,p}$ be given in \mathbf{R}^N . Then the corresponding capacity value after a projection been made to hyperplane (or any plane) is less than or to equal the first capacity value up to a constant factor.

– This is a result from non-linear potential theory.

We will assume the definition of Hausdorff measure to be known, at least as in our case when the defining function h_α is the α -power of the cubes side length.

We here give one of the well-known connections between non-linear capacity and Hausdorff measure, proved by non-linear potential theory.

Fact 2.20. (A special case.) Let $M \subset \mathbf{R}^n$ and $C_{1,p}(M) = 0$, then for $\epsilon > 0$ it holds that $h_{n-p+\epsilon} = 0$.

The fact below is immediate from the definitions.

Fact 2.21. Let M be as above, then

- (i) $\alpha < \beta$ and $h_\alpha(M) = 0$ then $h_\beta(M) = 0$;
- (ii) h_n and n -dimensional Lebesgue measure are the same.

We assume the concept singular to be known. The fact below is well-known and elementary.

Fact 2.22. A continuous function on a line is the unique sum of an absolutely continuous part and a singular part, i.e. direct sum in vector sense.

We proceed with the proof and choose to prove (2.10).

Since there always is the possibility of a partition of unity using smooth functions, it is clear that we can confine ourselves to the question for an arbitrary cube B , such that its closure lies in Ω .

Hence let $u \in Q \cap V^{1,p}(B)$. Furthermore let L_j be the set of lines in the x_j -direction with the property that u is continuous on each line in the set L_j . To state it clearly we use a restriction notation for classes of functions with respect to L_j and similar sets, but this shall be interpreted on an individual function basis, or formally as L_j etc. as dependent on the function involved.

– This is a minor change versus the normal notations.

Notation 2.23. Let

$$(2.16) \quad \begin{aligned} E_j &= \bigcup_{l \in L_j} l \\ E &= \cap_j E_j \end{aligned}$$

Now we want to show that

$$(2.17) \quad m_{N-1}(E_j^c \cap M_{N-1}) = 0$$

Here M_{N-1} is a $N-1$ dimensional hyperplane perpendicular to the x_j -direction and m_{N-1} is the $N-1$ dimensional Lebesgue measure.

To show this we start with that $u \in Q$. Then the points where u is discontinuous can be kept in an open set ω_ϵ such that $C_{1,p}(\omega_\epsilon) \leq \epsilon$. The set $D\omega_\epsilon$ can be projected orthogonally to M_{N-1} and we get the set $\omega_\epsilon^{M_{N-1}}$.

Now by Fact 2.19 we have

$$(2.18) \quad C_{1,p}(\omega_\epsilon^{M_{N-1}}) \leq A\epsilon$$

However we if go to the limit we really have

$$(2.19) \quad C_{1,p}(E_j^c \cap M_{N-1}) = 0.$$

Now by Fact 2.20 we have that for $\epsilon' > 0$ it holds that

$$(2.20) \quad h_{N-p+\epsilon'}(E_j^c \cap M_{N-1}) = 0.$$

This is rewritten as

$$(2.21) \quad h_{N-1-(p-1)+\epsilon'}(E_j^c \cap M_{N-1}) = 0.$$

Now if ϵ' is chosen less than $p - 1$ – or why not equal to – which is possible since we as always assume $p > 1$, then we have

$$(2.22) \quad h_{N-1}(E_j^c \cap M_{N-1}) = 0.$$

But by Fact 2.21 this implies that

$$m_{N-1}(E_j^c \cap M_{N-1}) = 0,$$

i.e. (2.17).

Now we are in a position to discuss (2.10) with domain the cube B . The LHS of (20) can be written as follows, as a consequence of Lemma 2.17,

$$(2.23) \quad \begin{aligned} & Q \cap V^{1,p}(B) / \sum_{j=1}^N [S_j \cap Q \cap V^{1,p}(B) / \check{X}_j \cap Q \cap V^{1,p}(B)] \\ &= \bigcap_{j=1}^N (Q \cap V^{1,p}(B) / [S_j \cap Q \cap V^{1,p}(B) / \check{X}_j \cap Q \cap V^{1,p}(B)]). \end{aligned}$$

However we should rather use some set L'_j instead of L_j , since the “almost every” prefix in the definitions of \check{X}_j etc. Take the intersection(s) and get L'_j .

This set obviously contains almost all lines in x_j -direction as well – a finite union of sets of measure zero has measure zero. Hence this adjustment has no significance. Furthermore redefine E_j to E'_j and E to E' .

Our next step is to set out to prove what is needed but with restriction to E' , i.e.

Step 2.24. It holds

$$\begin{aligned}
& \bigcap_{j=1}^N (Q \cap V^{1,p}(B) / [S_j \cap Q \cap V^{1,p}(B) / \check{X}_j \cap Q \cap V^{1,p}(B)])|_{E'} = \\
(2.24) \quad & = [AC] \cap Q \cap V^{1,p}(B)|_{E'}.
\end{aligned}$$

Now again we will return to the basis argument via Zorn's lemma and Venn diagramms used to prove Lemma 2.17. In this case we get the following

$$\begin{aligned}
& \bigcap_{j=1}^N Q \cap V^{1,p}(B) / [S_j \cap Q \cap V^{1,p}(B) / \check{X}_j \cap Q \cap V^{1,p}(B)]|_{E'} = \\
(2.25) \quad & = V^{1,p}(B) \cap \bigcap_{j=1}^N Q / [S_j \cap Q / \check{X}_j \cap Q]|_{E'}.
\end{aligned}$$

However it follows from the definitions of S_j and \check{X}_j as well as the unique decomposition of a continuous function on a line as a sum of an absolutely continuous part and a singular part (Fact 2.22), that

$$(2.26) \quad Q / [S_j \cap Q / \check{X}_j \cap Q]|_{E'_j} = [AC]_j \cap Q|_{E'_j}.$$

Now the information adds up. We use (2.24) and (2.25) together with the definition of $[AC]$ and get Step 2.24, i.e. (2.24).

This result Step 2.24 has because of the very definition of $[AC]$ the implication that a drop of the restriction gives

$$\begin{aligned}
& \bigcap_{j=1}^N (Q \cap V^{1,p}(B) / [S_j \cap Q \cap V^{1,p}(B) / \check{X}_j \cap Q \cap V^{1,p}(B)]) \subset \\
(2.27) \quad & \subset [AC] \cap Q \cap V^{1,p}(B).
\end{aligned}$$

Fact 2.25. It holds that

$$(2.28) \quad [AC] \cap V^{1,p}(\Omega) \subset W^{1,p}(\Omega); \text{ Here } \Omega = B$$

The LHS above was the original definition of first order Sobolev space. After Sobolev made his definition it has been more in fashion.

– There are other definitions still.

Now there is lining up of the inclusions

$$\begin{aligned}
 (2.29) \quad Q \cap W^{1,p}(B) &\subset Q \cap V^{1,p}(B) / \sum_{j=1}^N [S_j \cap Q \cap V^{1,p}(B) / \check{X}_j \cap Q \cap V^{1,p}(B)] \\
 &\subset Q \cap [AC] \cap V^{1,p}(B) \subset Q \cap W^{1,p}(B).
 \end{aligned}$$

Since the begin equals the end, all inclusions actually are equalities.

Hence we have proved (2.10), i.e. the theorem.

End of proof.

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